MS(QMS)
2019

1. (a) Find the value of $\lim_{n\to\infty}(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{3n})$.

   (b) Show that every square matrix is uniquely expressible as the sum of a symmetric and a skew symmetric matrix. [7 + 8 = 15]

2. (a) Let x be a positive real number. Then show that for any x, $x^2+\pi^2+x^{2\pi} \geq x^{\pi}(\pi+x)+x\pi$.

   (b) Solve the differential equation $\frac{dy}{dx}=x-xy$. [10 + 5 = 15]

3. (a) Let x be chosen at random from the interval $(0,1)$. What is the probability that $[\log_{10} 4x]-[\log_{10} x]=0$? Here $[x]$ denotes the greatest integer that is less than or equal to x.

   (b) The graph of $2x^2+xy+3y^2-11x-20y+40=0$ is an ellipse in the first quadrant of the $xy-$plane. Let a and b be the maximum and minimum values of y/x over all the points (x,y) on the ellipse. Find the value of $a+b$. [8 + 7 = 15]

4. (a) Let $f(x)$ be the determinant of the following matrix:

$$\begin{bmatrix} a^2x+1 & (b^2-1)x & (c^2-1)x \\ (a^2-1)x & b^2x+1 & (c^2-1)x \\ (a^2-1)x & (b^2-1)x & c^2x+1 \end{bmatrix}.$$

   If $a^2+b^2+c^2=2$, then what is the degree of the polynomial $f(x)$?

   (b) Let $g(x)=x^6-x^5+x^2-x+3, -\infty<x<\infty$. Show that $g(x)>0$ for all x. [7 + 8 = 15]

5. (a) Evaluate $\int\int\sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}}dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

   (b) Let $f(x)$ and $g(x)$ are differentiable functions for $0\leq x\leq 1$ such that $f(0)=2, g(0)=0, f(1)=6$ and $g(1)=2$. Then show that there exists c satisfying $0<c<1$ and $f'(c)=2g'(c)$.

   [9 + 6 = 15]

6. (a) Let $\{x_n : n=0,1,2,\ldots\}$ be a sequence of real numbers such that $x_{n+1}=\lambda x_n+(1-\lambda)x_{n-1}$ for $n\geq 1$ and for some $\lambda, 0<\lambda<1$.

      i. Show that $x_n = x_0+(x_1-x_0)\sum_{k=0}^{n-1}(\lambda-1)^k$

      ii. Hence or otherwise show that x_n converges as $n\to\infty$ and find the limit.

   (b) Show that the height of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h is $\frac{h}{3}$. [(6 + 3) + 6 = 15]

7. (a) Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$. Then show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)dx = 1 - \frac{1}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{2}\log 2$.

(b) Let $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ be such that $\min f(x) > \max g(x)$. Show that $|c| > \sqrt{2}|b|$. [8 + 7 = 15]

8. (a) A piece of cheese is located at $(12, 10)$ in a co-ordinate plane. A mouse is at $(4, -2)$ and is running up the line $y = -5x + 18$. At the point (a, b) the mouse starts getting farther from the cheese rather than closer to it. What is $a + b$?

(b) Let $S_1 = \{(x, y) | \log_{10}(1 + x^2 + y^2) \leq 1 + \log_{10}(x + y)\}$ and $S_2 = \{(x, y) | \log_{10}(2 + x^2 + y^2) \leq 2 + \log_{10}(x + y)\}$. What is the ratio of the area of S_2 to the area of S_1? [5 + 10 = 15]